

# OPEN SETS OF AXIOM A FLOWS WITH EXPONENTIALLY MIXING ATTRACTORS

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**ABSTRACT.** For any dimension  $d \geq 3$  we construct  $C^1$ -open subsets of the space of  $C^3$  vector fields such that the flow associated to each vector field is Axiom A and exhibits a non-trivial attractor which mixes exponentially with respect to the unique SRB measure.

## 1. INTRODUCTION

The Axiom A flows of Smale [21] have been extensively studied in the last three decades and are now relatively well understood. One important remaining question concerns the rate of mixing for such flows. The conjecture that all mixing Axiom A flows mix exponentially was quickly proven false by Ruelle [20] and then Pollicott [18] constructed examples which mix arbitrarily slowly. Building on work by Chernov [8], Dolgopyat [9] demonstrated that all mixing Anosov flows with  $C^1$  stable and unstable foliations mix exponentially. Unfortunately this good regularity of the invariant foliations is not typical [14]. Dolgopyat also showed [10] that rapid mixing (super polynomial) is typical, in a measure theoretic sense of prevalence, for Axiom A flows (mixing is with respect to any equilibrium state associated to a Hölder potential). Building on these ideas Field, Melbourne, and Török [12] showed that there exist  $C^2$ -open,  $C^r$ -dense sets of Axiom A flows for which each non-trivial basic set is rapid mixing. It is tempting to think that exponential mixing is a robust property, i.e., if an Axiom A flow mixes exponentially then all sufficiently close Axiom A flows also mix exponentially. This remains an open problem, even limited to the case of Anosov flows. Dolgopyat [11] conjectured that the set of exponentially mixing flows contains a  $C^r$ -open and dense subset of the set of all Axiom A flows. For a more complete history of the question of rates of mixing of hyperbolic flows the reader is directed towards the introductions of [16, 12].

The purpose of this short article is to construct open sets of Axiom A flows which mix exponentially. The first and third author previously constructed [3] open sets of three-dimensional singular hyperbolic flows (geometric Lorenz attractors) which mix exponentially with respect to the unique SRB measure. The question was raised that perhaps the presence of the singularity in these flows actually aids the mixing and allows for the robust exponential mixing. In this article we show that we do

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*Date:* February 27, 2014.

2010 *Mathematics Subject Classification.* Primary: 37D20, 37A25; Secondary: 37C10.

*Key words and phrases.* Robust exponential decay of correlations, SRB measure, Axiom A flows.

It is a pleasure to thank Ian Melbourne for helpful advice. O.B. is grateful to Henk Bruin for several discussions. O.B. acknowledges the support of the Austrian Science Fund, Lise Meitner position M1583. P.V. and V.A. were partially supported by CNPq, PRONEX-Dyn.Syst. and FAPESB (Brazil).

not need the presence of the singularities but actually we can just take advantage of the volume contraction of the flow (and consequently a domination condition) in order to carry out a similar construction.

## 2. RESULTS & OUTLINE OF THE PROOF

Let  $M$  be a Riemannian manifold and  $X^t : M \rightarrow M$  the flow associated to the vector field  $X$ . We consider the  $\mathcal{C}^r$  distance on the space of  $\mathcal{C}^r$ -vector fields  $\mathfrak{X}^r(M)$ . Note that this induces a natural distance on the space of flows. We say that the flow is  $\mathcal{C}^r$  if the associated vector field is  $\mathcal{C}^r$ . We recall the notion of a (hyperbolic) basic set [7]. In particular a basic set  $\Lambda \subset M$  is closed, invariant, hyperbolic, and  $X^t|_\Lambda$  is topologically transitive. A basic set is called an attractor if there exists an open set (the trapping region)  $U \subset M$ , and  $t_0 > 0$ , such that  $X^{t_0}U \subset U$ . Consequently  $\Lambda = \bigcap_{t \in \mathbb{R}_+} X^t U$ . A basic set is *non-trivial* if it is neither an equilibrium nor a periodic solution. It is known that for every basic set of an Axiom A flow there exists a unique SRB measure which we denote by  $\mu$ . For any Hölder continuous observables  $\phi, \psi : U \rightarrow \mathbb{R}$  we define the correlation

$$\rho_{\phi, \psi}(t) := \int \phi(X^t x) \cdot \psi(x) \, d\mu - \int \phi(x) \, d\mu \cdot \int \psi(x) \, d\mu,$$

$t \geq 0$ . We say that the flow mixes exponentially if there exists  $\gamma > 0$  and for each  $\phi, \psi$  (Hölder on  $U$ ) there exists  $C_{\phi, \psi}$  such that  $|\rho_{\phi, \psi}(t)| \leq C_{\phi, \psi} e^{-\gamma t}$  for all  $t \geq 0$ . The following two theorems are the main results of this article.

**Theorem A.** *Given any compact Riemannian manifold  $M$  of dimension  $d \geq 3$  there exists a  $\mathcal{C}^1$ -open subset of  $\mathcal{C}^3$ -vector fields  $\mathcal{U} \subset \mathfrak{X}^3(M)$  such that for each  $X \in \mathcal{U}$  the associated flow is Axiom A and exhibits a non-trivial attractor which mixes exponentially with respect to the unique SRB measure.*

As far as the authors are aware, this is the first published result concerning the existence of robustly exponentially mixing Axiom A flows. The strategy for the construction of the open sets in the above theorem is similar to the one developed in [3] for singular flows. Theorem A is a consequence (details in Section 4) of the following more fundamental result.

**Theorem B.** *Suppose that  $X^t : M \rightarrow M$  is a  $\mathcal{C}^2$  Axiom A flow,  $M$  is of dimension  $d \geq 3$ , and  $\Lambda$  is an attractor. Further suppose that the stable foliation is  $\mathcal{C}^2$ . Then either the flow mixes exponentially with respect to the unique SRB measure or there exists an  $X^t$ -invariant foliation of  $U \supset \Lambda$  which is  $(d-1)$ -dimensional and transverse to the flow direction.*

The proof of the above is described in Section 3 and involves quotienting along stable manifolds of a well-chosen Poincaré section to reduce to the case of a suspension semiflow over an expanding Markov map. We can then apply the result of Avila, Gouëzel, and Yoccoz [4] which implies exponential mixing for the semiflow unless the return time function is cohomologous to a piecewise constant function. This then gives exponential mixing for the original flow. We have no reason to believe that the requirement of a  $\mathcal{C}^2$  stable foliation is essential to the above theorem, however the present methods rely heavily on this fact. Note that although we require this good regularity of the stable foliation we have no requirements on the regularity of the unstable foliation. This is in contrast to Dolgopyat's original argument which required  $\mathcal{C}^1$  regularity for both the stable and unstable foliation.

The following questions remain: Are all exponentially mixing Axiom A flows also robustly exponentially mixing? Is the set of exponential mixing flows dense among the set of all Axiom A flows? It would appear that both these questions are of a higher order of difficulty.

The second alternative in Theorem B can be seen in several different ways. It is equivalent to the flow being a suspension over an Axiom A diffeomorphism where the return time function is constant on each element of the Markov partition. In this case it is possible that the flow does not even mix or may mix arbitrarily slowly. It may also be seen as the joint integrability of the stable and unstable foliations.

The additional ingredient in order to use Theorem B in order to prove Theorem A is a result concerning the regularity of foliations. First we must introduce a little more notation. Let  $\|\cdot\|$  denote the Riemannian norm on the tangent space of  $M$ . As before  $X^t : M \rightarrow M$  is an Axiom A flow and  $\Lambda$  is a non-trivial basic set. Since the flow is Axiom A the tangent bundle restricted to  $\Lambda$  can be written as the sum of three  $DX^t$ -invariant continuous subbundles,  $T_\Lambda M = \mathbb{E}^s \oplus \mathbb{E}^c \oplus \mathbb{E}^u$  where  $\mathbb{E}^c$  is the one-dimensional bundle tangent to the flow and there exists  $C, \lambda > 0$  such that  $\|DX^t|_{\mathbb{E}^s}\| \leq Ce^{-\lambda t}$ , and  $\|DX^{-t}|_{\mathbb{E}^u}\| \leq Ce^{-\lambda t}$ , for all  $t \geq 0$ .

**Theorem 1.** *Suppose that  $\Lambda$  is an attractor for the  $\mathcal{C}^3$  Axiom A flow  $X^t : M \rightarrow M$ . Further suppose that*

$$\sup_{x \in \Lambda} \|DX_x^{t_0}|_{\mathbb{E}^s}\| \cdot \|DX_x^{t_0}\|^2 < 1, \quad (1)$$

*for some  $t_0 > 0$ . Then the stable foliation of  $X^t$  is  $\mathcal{C}^2$ .*

This is a classical result of the arguments<sup>1</sup> described by Hirsh, Pugh, and Shub [15].

As hinted earlier this question of regularity is a subtle issue. In the case of Anosov flows, one cannot in general expect the stable foliation (or equivalently the unstable foliation) to be better than Hölder [14]. If the invariant foliation of interest is codimension one then the argument can be improved and better regularity obtained. However the stable foliation of a hyperbolic flow can never be codimension one. Another case where a better argument is possible is when the flow preserves a contact form. In this case it is possible to show that the stable foliation for a three-dimensional contact Anosov flow is better than  $\mathcal{C}^1$ . For higher dimensions Liverani [16] was able to bypass the regularity of the stable and unstable invariant foliations and show exponential mixing using the contact structure, an observation that suggests that the regularity of the stable foliation is not essential.

We will construct open sets of Axiom A flows which satisfy the assumptions of Theorem 1. We remind the reader that such flows must be dissipative, therefore the hyperbolic basic set  $\Lambda$  is of zero volume, and thus such flows are necessarily not Anosov. Moreover, since the domination condition of Theorem 1 is an open condition it implies that there exist open sets of Axiom A flows which possess a  $\mathcal{C}^2$  stable foliation. Therefore, to prove that robust exponential mixing does exist we apply Theorem B and prove that the non-existence of an  $X^t$ -invariant  $(d-1)$ -dimensional transverse foliation is also an open condition. Details of this argument are given in Section 4.

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<sup>1</sup>Actually Theorem 1 holds in greater generality. We may replace the exponent “2” by any real number  $k > 0$ . In this case the stable foliation is  $\lfloor k \rfloor$ -times differentiable with derivative that is  $(k - \lfloor k \rfloor)$ -Hölder.

### 3. THE DICHOTOMY FOR AXIOM A ATTRACTORS WITH $C^2$ -STABLE FOLIATION.

The purpose of this section is to prove Theorem B. For the duration of this section we fix  $X \in \mathfrak{X}^2(M)$  where  $M$  is  $d$ -dimensional and assume that  $X^t : M \rightarrow M$  is a  $C^2$  Axiom A flow and that  $\Lambda$  is an attractor with a  $C^2$  stable foliation. In the following, the terms *Markov family*, *local section*, and *rectangle* are as defined by Bowen [6]. The first step is to carefully choose a Poincaré section for the flow.

**Lemma 2.** *There exists a Markov family  $\mathcal{R} = \{\mathcal{R}_i\}_i$  consisting of a finite number of rectangles  $\mathcal{R}_i \subset \Lambda$ . There exist  $(d-1)$ -dimensional  $C^2$  hypersurfaces  $\mathcal{S} = \{\mathcal{S}_i\}_i \subset U$  transverse to the flow such that  $\mathcal{R}_i = \mathcal{S}_i \cap \Lambda$ . Each rectangle  $\mathcal{R}_i$  has the product structure  $[\Delta_i, \Gamma_i]$  where the  $\Delta_i$  are  $C^2$  disks which are contained within local unstable manifolds and the  $\Gamma_i$  are subsets of the local stable manifolds.*

*Proof.* By [6, Theorem 2.5] we know that  $X^t : \Lambda \rightarrow \Lambda$  has Markov families of local sections of arbitrarily small size. However we need to modify this Markov family so the hypersurfaces associated with the rectangles are  $C^2$  and foliated by local stable manifolds. Suppose that  $\mathcal{R}_i$  is one of the rectangles of the Markov family and that  $\mathcal{S}_i \supset \mathcal{R}_i$  is the corresponding hypersurface.

Given any  $\varepsilon > 0$  and a continuous function  $g : \mathcal{S}_i \rightarrow [-\varepsilon, \varepsilon]$  we define a new rectangle  $\mathcal{R}'_i = \{X^{g(x)}(x) : x \in \mathcal{R}_i\}$  and also define similarly  $\mathcal{S}'_i = \{X^{g(x)}(x) : x \in \mathcal{S}_i\}$ . These sets are clearly contained in the open set  $B_{i,\varepsilon} = \{X^t(x) : x \in \mathcal{S}_i, -\varepsilon \leq t \leq \varepsilon\}$ . As long as  $\varepsilon$  is sufficiently small then the Markov structure of the family is preserved. We use this idea to slightly modify the original Markov family so that the requirements of the lemma hold. The construction can be done as follows. Let  $\Delta'_i$  be a  $C^2$  smoothed version of  $\Delta_i$ . Then consider the section produced by the union of local stable manifolds which intersect  $\Delta'_i$ , that is

$$\mathcal{S}'_i = \bigcup_{x \in \Delta'_i} W^{ss}(x) \cap B_{i,\varepsilon}. \quad (2)$$

Since the stable foliation of  $X^t : \Lambda \rightarrow \Lambda$  is  $C^2$  then each of the sections  $\mathcal{S}'_i$  is  $C^2$  as it is foliated by  $C^2$  local stable manifolds. We use the fact that  $\Lambda$  is an attractor to ensure that the  $\Delta_i$  of the product structure are  $C^2$  disks and not Cantor-like sets.  $\square$

Now we have such a Poincaré section  $\mathcal{S} = \{\mathcal{S}_i\}_i \subset U$  we can represent the flow as a suspension flow. We denote the return map by  $P : \mathcal{S} \rightarrow \mathcal{S}$  and the return time by  $\tau : \mathcal{S} \rightarrow \mathbb{R}_+$ . Let  $\mathcal{F}_s$  denote the foliation of  $U$  by local stable manifolds. A key point in this construction is that the return time  $\tau$  is constant along the local stable manifolds and  $C^2$ -smooth. Quotienting along these manifolds we obtain a suspension semiflow over an expanding map. The fact that the section is foliated by local stable manifolds is essential. Let  $\Delta = \mathcal{S} / \mathcal{F}_s$  (the quotient of  $\mathcal{S}$  with respect to the local stable manifolds). A concrete realization of this quotient is given by  $\Delta = \{\Delta_i\}_i$ . Let  $F : \Delta \rightarrow \Delta$  denote the quotiented return map and for future convenience let  $\pi : \mathcal{S} \rightarrow \Delta$  denote the corresponding projection. Since the return time is constant along stable leaves we also write, by some abuse of notation, the return time  $\tau : \Delta \rightarrow \mathbb{R}_+$ . Denote by  $\mathbf{m}$  the induced normalised Riemannian volume form.

In the following we will make the connection to the flows studied in [4]. It will turn out that our setting is simpler in several ways, including that fact that the Markov partition is finite and not merely countable, the return time  $\tau : \Delta \rightarrow \mathbb{R}_+$

is uniformly bounded, both from above and from below, and the domain of the expanding Markov map is merely the union of disks. Let  $d_u$  denote the dimension of the unstable bundle  $\mathbb{E}^u$ , and let  $d_s$  denote the dimension of the stable bundle  $\mathbb{E}^s$ .

**Lemma 3.** *The map  $F : \Delta \rightarrow \Delta$  is a  $C^2$  uniformly expanding Markov map, i.e.,*

- (1)  $\Delta = \{\Delta_i\}_i$  is the disjoint finite union of  $d_u$ -dimensional  $C^2$  disks,
- (2) For each  $i$  there exists a finite partition  $\{\Delta_{i,\ell}\}_\ell$  (**m** modulo zero) of  $\Delta_i$  by open subsets so that  $F : \Delta_{i,\ell} \rightarrow \Delta_\ell$  is a  $C^2$  diffeomorphism for every  $i, \ell$ ;
- (3) There are  $C > 0$  and  $\lambda \in (0, 1)$  such that  $\|(DF^n)^{-1}\| \leq C\lambda^n$ .

*Proof.* By Lemma 2 each disconnected component of the Poincaré section for the original flow has the appropriate product structure which implies the Markov structure of  $F : \Delta \rightarrow \Delta$ . The  $C^2$ -smooth regularity of the cross-sections  $\mathcal{S}_i$  and the flow  $(X^t)_t$  is enough to guarantee that the return map  $P : \mathcal{S} \rightarrow \mathcal{S}$  is also  $C^2$  on each component. Using that the stable holonomy is also  $C^2$ -smooth, this in turn implies that  $F : \Delta_{i,\ell} \rightarrow \Delta_\ell$  is a  $C^2$  diffeomorphism when restricted to each partition element. This is enough to obtain properties (1) and (2). The uniform expansion estimates follow from the uniform hyperbolicity of the original flow. In fact, since the diameter of the Markov structure can be taken arbitrarily small we assume this is the case so that the return time function  $\tau$  has uniform bounds  $0 < \tau_0 \leq \tau(\cdot) \leq \tau_1 < \infty$  satisfying  $\lambda_1 := CK\lambda^{\tau_0} < 1$ , where  $K$  is a uniform bound for the derivative  $D\pi$  of the holonomy function  $\pi$  on  $\mathcal{S}$ . Then, the property that  $\|(DX_t|_{E^u})^{-1}\| \leq C\lambda^t$  applied to the  $n$ th return to the cross section yields that  $\|(DF^n)^{-1}\| \leq \lambda_1^n$ . This proves (3) and completes the proof of the lemma.  $\square$

**Lemma 4.** *The return map  $P : \mathcal{S} \rightarrow \mathcal{S}$  is a hyperbolic skew-product [4, Definition 2.5].*

*Proof.* We must show that the following properties hold:

- (1)  $\pi : \mathcal{S} \rightarrow \Delta$  is continuous and  $F \circ \pi = \pi \circ P$  whenever both members of the equality are defined;
- (2) there is a  $P$ -invariant probability measure  $\eta$  giving full mass to  $\mathcal{S}$ ;
- (3) there exists a family of probability measures  $\{\eta_x\}_{x \in \Delta}$  on  $\mathcal{S}$  which is a disintegration of  $\eta$  over  $\nu$ , that is,  $x \mapsto \eta_x$  is measurable,  $\eta_x$  is supported on  $\pi^{-1}(x)$  and, for each measurable subset  $A$  of  $\mathcal{S}$  we have  $\eta(A) = \int \eta_x(A) d\nu(x)$ . Moreover, this disintegration is smooth: we can find a constant  $C > 0$  such that, for any open subset  $V \subset \bigcup \Delta_{i,\ell}$  and for each  $u \in C^1(\pi^{-1}(V))$ , the function  $\tilde{u} : V \rightarrow \mathbb{R}, x \mapsto \tilde{u}(x) := \int u(y) d\eta_x(y)$  belongs to  $C^1(V)$  and satisfies

$$\sup_{x \in V} \|D\tilde{u}(x)\| \leq C \sup_{y \in \pi^{-1}(V)} \|Du(y)\|.$$

- (4) there is  $\kappa \in (0, 1)$  such that, for all  $w_1, w_2 \in \mathcal{S}_i$  in the same local stable leaf, i.e.  $\pi(w_1) = \pi(w_2)$ , we have  $d(Pw_1, Pw_2) \leq \kappa d(w_1, w_2)$ .

Property (3) can be established similarly as in [3] for the geometric Lorenz attractor while the other properties are clear. It may happen that property (4) only holds for some iterate of the map. In this case we simply use the standard technique of an *adapted metric* (see [13]) to obtain the above required estimate with respect to the new distance.  $\square$

We define the suspension flow in the usual way: Let  $\mathcal{S}_\tau = \{(x, u) : x \in \pi^{-1}(\Delta), 0 \leq u < \tau(\pi x)\}$ . Define the flow  $(P)_t : \mathcal{S}_\tau \rightarrow \mathcal{S}_\tau$  by  $(P)_t(x, u) = (x, u + t)$  whenever  $u + t < \tau(\pi x)$  and by  $(P)_t(x, u) = (P(x), 0)$  whenever  $u + t = \tau(\pi x)$ . The flow is defined for all  $t \geq 0$  by assuming that the semigroup property  $(P)_{t+s} = (P)_t \circ (P)_s$  continues to hold. Note that the suspension flow  $(P)_t : \mathcal{S}_\tau \rightarrow \mathcal{S}_\tau$  is intimately connected to the original flow  $X^t : U \rightarrow U$  for some neighbourhood  $U$  of the attractor.

We say that  $\tau : \Delta \rightarrow \mathbb{R}_+$  is *cohomologous to a piecewise constant function* if there exists some  $\mathcal{C}^1$  function  $\gamma : \bigcup_{i,\ell} \Delta_{i,\ell} \rightarrow \mathbb{R}$  such that  $\tau - \gamma \circ F + \gamma$  is constant on each  $\Delta_{i,\ell}$ . Following [4, Definition 2.6], the suspension flow  $(P)_t : \mathcal{S}_\tau \rightarrow \mathcal{S}_\tau$  over the hyperbolic skew-product  $P$  is an *excellent hyperbolic semiflow* if  $\tau$  is bounded from below by some positive constant  $\tau_0$ ,  $\tau$  is not cohomologous to a piecewise constant function and there exists a positive constant  $K$  so that  $|D(\tau \circ h)| \leq K$  for every inverse branch  $h$  of  $F$ .

**Lemma 5.** *Suppose that  $\tau : \Delta \rightarrow \mathbb{R}_+$  is not cohomologous to a piecewise constant function. Then  $(P)_t : \mathcal{S}_\tau \rightarrow \mathcal{S}_\tau$  is an excellent hyperbolic semiflow.*

*Proof.* This follows immediately from Lemma 4, the supposition concerning not being cohomologous, and the definition of an excellent hyperbolic semiflow.  $\square$

That  $(P)_t : \mathcal{S}_\tau \rightarrow \mathcal{S}_\tau$  is an excellent hyperbolic semiflow means [4, Theorem 2.7] that it mixes exponentially for  $\mathcal{C}^1$  observables. This in turn means that the original flow  $X^t : U \rightarrow U$  also mixes exponentially for  $\mathcal{C}^1$  observables. By the usual approximation argument this implies exponential mixing for Hölder observables (for some slower, yet still exponential, rate)<sup>2</sup>. To complete the proof of Theorem B it remains to show the other alternative to the supposition of the above lemma.

**Lemma 6.** *Suppose that  $\tau : \Delta \rightarrow \mathbb{R}_+$  is cohomologous to a piecewise constant function. Then there exists a  $X^t$ -invariant foliation  $\mathcal{F}^{su}$  of  $U$  which is  $(d-1)$ -dimensional and transverse to the flow direction.*

*Proof.* We will modify the Poincaré section  $\mathcal{S}$  and take advantage of the cohomology equation  $\tau - \gamma \circ F + \gamma = \chi$ , where  $\chi$  is constant on each  $\Delta_{i,\ell}$ . We will do this modification in such a way that we preserve the Markov structure and such that the return time function of the suspension flow associated to the new section is piecewise constant. The partition of  $\Delta$  naturally gives a partition of  $\mathcal{S}$ , i.e.,  $\mathcal{S}_{i,\ell} = \pi^{-1}(\Delta_{i,\ell})$ . For each  $i, \ell$  fix some  $y_{i,\ell} \in \Delta_{i,\ell}$ . Define the components of the new section by

$$\mathcal{S}_{i,\ell} = \{X^r x : x \in \mathcal{S}_{i,\ell} \text{ and } r = \gamma(\pi x) - \gamma(y_{i,\ell})\}.$$

This is similar to the proof of Lemma 2. It may happen that the components of the new section are not disjoint as subsets of  $M$ . In this case we artificially subdivide so that each of the original components of the section is small. We can do this simply using a refinement of the Markov partition. Note that the return time for the new section, at a point  $x \in \mathcal{S}_{i,\ell}$ , is given by the expression  $\tau(\pi x) - \gamma(y_{i,\ell}) + \gamma(\pi x) + \gamma(F(y_{i,\ell})) - \gamma(F(\pi x))$  which, by the cohomology equation, is constant on each  $\mathcal{S}'_{i,\ell}$ .  $\square$

<sup>2</sup>One uses the following fact concerning Hölder functions: If  $f \in \mathcal{C}^\alpha$  then there exists  $C > 0$  such that for all  $\varepsilon > 0$  there exists  $f_\varepsilon \in \mathcal{C}^1$  such that  $\|f - f_\varepsilon\|_{\mathcal{C}^0} \leq \varepsilon C$  and  $\|f_\varepsilon\|_{\mathcal{C}^1} \leq \varepsilon^{-(1-\alpha)}$ . This approximation can be applied to the two observables in the definition of correlation with  $\varepsilon$  chosen as a function of  $t$ , smaller as  $t$  becomes larger.



## 4. ROBUST EXPONENTIAL MIXING

The purpose of this section is to prove Theorem A. We will make use of the dichotomy in Theorem B and the regularity of the foliation given by Theorem 1. Let  $M$  be a Riemannian manifold of dimension  $d \geq 3$ . It is sufficient to prove that there exists an open set of vector fields supported on the  $d$ -dimensional hypercube  $D^d := (0, 1)^d$  satisfying the conclusion of Theorem A.

**Lemma 7.** *For any  $d \geq 3$  there exists a vector field  $X \in \mathfrak{X}^3(\mathbb{R}^d)$  such that  $X^t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  exhibits an Axiom A attractor contained within  $(0, 1)^d$ . Moreover the domination condition (1) holds.*

*Proof.* The *Plykin attractor* [19, §8.9] is a smooth diffeomorphism of a bounded subset of  $\mathbb{R}^2$  which exhibits an Axiom A attractor. By [17, §2] we may use the existence of an Axiom A diffeomorphism on  $D^2$  to construct an Axiom A flow exhibiting an attractor in a bounded subset of  $D^2 \times S^1$ . This can be embedded in  $\mathbb{R}^3$  as a solid torus. It is a simple matter to ensure that the contraction in the stable direction is much stronger than the expansion in the unstable direction so that the domination condition (1) holds. An attractor of higher dimensions is achieved by a similar construction but with additional uniformly contracting directions added.  $\square$

The above construction is far from being the only possibility. To construct a four dimensional flow another obvious choice is to start with the *Smale Solenoid Attractor*. This is an Axiom A diffeomorphism in  $\mathbb{R}^n$  which exhibits an attractor  $\Lambda$  with one-dimensional unstable foliation and  $n - 1$ -dimensional stable foliation. Constructing a flow from this as per the above proof gives a  $(n + 1)$ -dimensional Axiom A flow exhibiting an attractor. For a wealth of possibilities in higher dimension we may take advantage of the work of Williams [22] on expanding attractors. By this the expanding part of the system is determined by a symbolic system of an  $n$ -solenoid, this means that the expanding part of the system may be any dimension desired. Then he shows that the stable directions may be added and the whole system embedded as a vector field on  $\mathbb{R}^d$ .

Since the domination condition (1) is open we obtain the following as an immediate consequence Lemma 7 and of the regularity given by Theorem 1: There exists a  $C^1$ -open subset  $\mathcal{U} \subset \mathfrak{X}^2(M)$  of Axiom A flows exhibiting a transitive attractor  $\Lambda$  with  $C^2$  stable foliation. Then, using Theorem B, for any  $X \in \mathcal{U}$  the corresponding transitive attractor for the Axiom A flow  $X^t : M \rightarrow M$  admits a  $C^2$  stable foliation and, consequently, either the unique SRB measure mixes exponentially or there exists a  $X^t$ -invariant foliation of a neighborhood of  $\Lambda$  in  $M$  which is  $(d - 1)$ -dimensional and transverse to the flow direction. Hence, to finish the proof of Theorem A we need to show that we can ensure that the second alternative is a closed condition and that the complement is not empty.

**Lemma 8.** *The construction of a vector field  $X \in \mathfrak{X}^3(M)$  according to Lemma 7 may be modified to ensure that there does not exist a  $(d - 1)$ -dimensional  $X^t$ -invariant foliation.*

*Proof.* It suffices to show that the construction can be modified such that  $\tau : \Delta \rightarrow \mathbb{R}_+$  is not cohomologous to a piecewise constant function. Choose two distinct periodic points  $x_1, x_2$  of  $F : \Delta \rightarrow \Delta$  of the same period  $n$  whose orbits are distinct. Furthermore ensure that each orbit visits each of the elements of the

Markov partition the same number of times as the other (but necessarily in some different order to each other). This means that the symbols in the first  $n$  coordinates of the itineraries of  $x_1$  and  $x_2$  in the shift space are the same. That such distinct orbits exist is due to the Markov structure and the connection to symbolic dynamics. For instance, in the case where the Markov partition has the two components  $\Delta_1, \Delta_2$  we can choose  $n = 4$  and  $x_1$  such that  $x_1, F(x_1) \in \Delta_1, F^2(x_1), F^3(x_1) \in \Delta_2$  and, on the other hand,  $x_2$  such that  $x_2, F^2(x_2) \in \Delta_1, F(x_2), F^3(x_2) \in \Delta_2$ .

Set  $\tau_n(x) = \sum_{j=0}^{n-1} \tau(F^j(x))$ . Since  $x_1$  and  $x_2$  visit the same Markov partition elements an equal number of times, if  $\tau : \Delta \rightarrow \mathbb{R}_+$  is cohomologous to a piecewise constant function then  $\tau_n(x_1) = \tau_n(x_2)$ . This means that we merely need to slightly alter the function  $\tau$  in a small neighbourhood of  $x_1$  that does not intersect the orbit of  $x_2$  to ensure that  $\tau : \Delta \rightarrow \mathbb{R}_+$  is not cohomologous to a piecewise constant function.  $\square$

*Remark 9.* The previous argument actually proves that for  $F : \Delta \rightarrow \Delta$  fixed there exists an open and dense set of continuous roof functions  $\tau : \Delta \rightarrow \mathbb{R}_+$  that are not cohomologous to a piecewise constant roof function.

We have now shown that there exist the above examples of flows which do not admit a codimension one invariant foliation transversal to the flow direction. Therefore, in order to complete the proof of Theorem A, it only remains to show that this property is open.

**Lemma 10.** *The condition that a flow does not admit a codimension one invariant foliation transversal to the flow direction is  $C^1$ -open in the space of Axiom A flows.*

*Proof.* It is convenient to recall the *temporal distance function* [10, § 5] (see also [16, Appendix A]). It is clear from the definition of this function that to prove the lemma it suffices to show that it is not identically zero for an open set of flows. By [12, Proposition 3.2] we know that the temporal distance function depends continuously on the flow (in the  $C^1$  topology on the space of Axiom A flows). Consequently the property that the temporal distance function is not identically zero is an open property.  $\square$

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